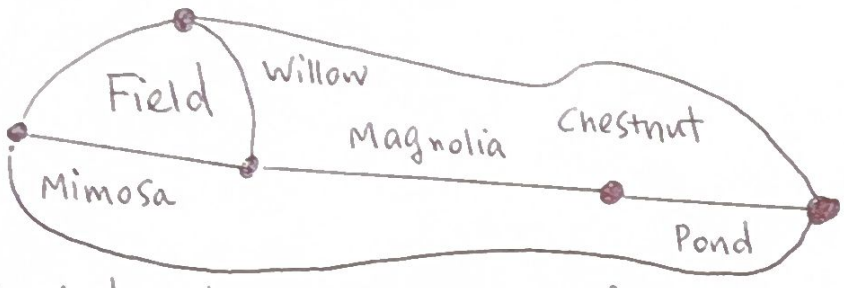
18 – Euler walks and circuits

To make campus safer, security will patrol dorms. How to walk each path (edge) exactly once & come back to start?

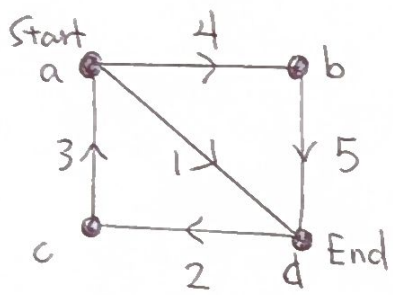


Defn A **walk v to w** on graph $G=(V, E)$ is a sequence of vertices $v_1=v, v_2, v_3, \dots, v_n=w$ of vertices such that

$$\{v_i, v_{i+1}\} \in E \quad \text{for} \quad 0 \leq i < n.$$

It's an **Euler walk** if these $\{v_i, v_{i+1}\}$ exhaust all edges of G.

Ex 1



Euler walk from a to d.

Defn A graph is **connected** if there is a walk between any two vertices.

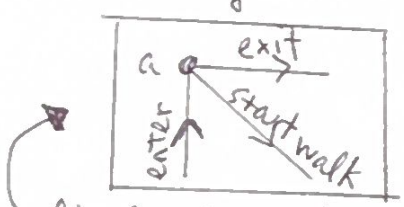
Ex of connected: 

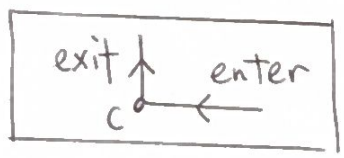
Not connected: 

Thm A connected graph has an Euler walk from v to w with $v \neq w$

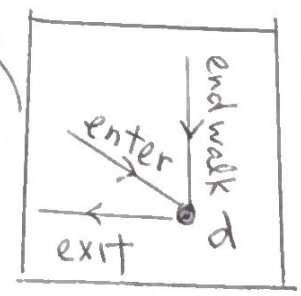
$\iff$ v and w are the only odd deg vertices.

Why? $\Rightarrow$ Look at Ex 1: at "pass-thru" vertex c, its deg is even since half the deg is to enter c & half to exit c, forming an enter-exit edge pair.



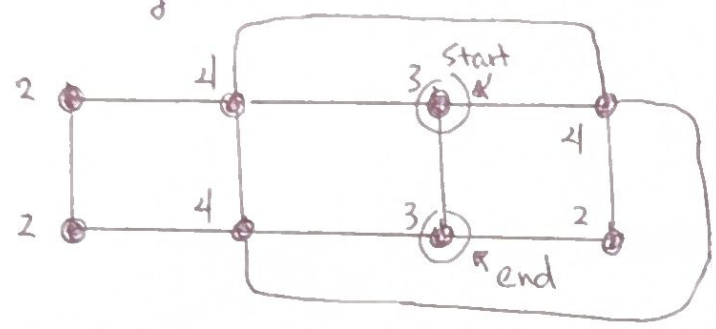


At start vertex a, one edge starts the walk, & rest are enter-exit edge pairs, so deg(a) = odd. Similarly for end vertex d.

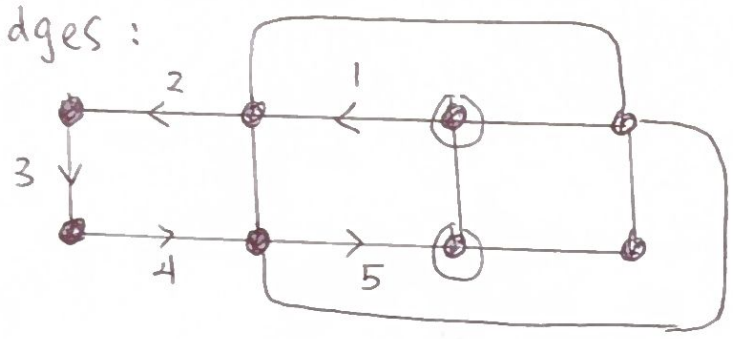


Proof ⇐ (converse) We use/introduce Hierholzer's algorithm:

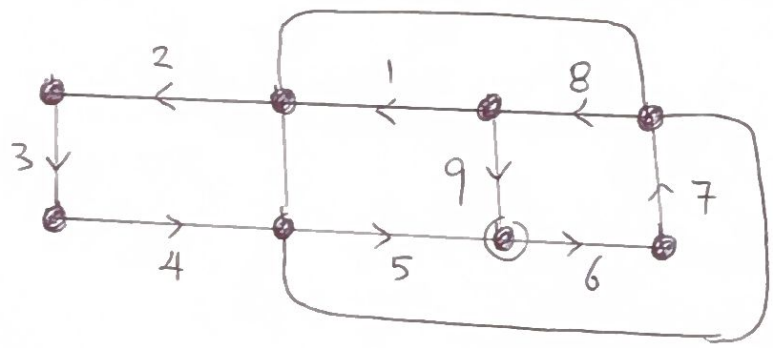
First find the 2 odd deg vertices to be the start & end vertices:



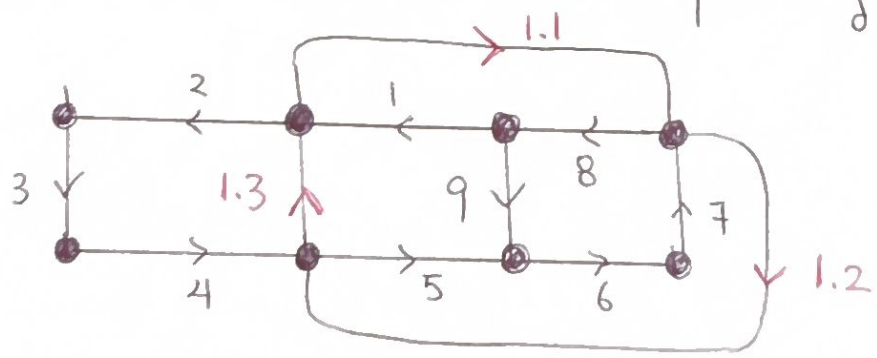
Then find (a possibly imperfect) walk from start to end, with some (but possibly not all) edges:



Since we can continue, we walk more edges until we return to end vertex again:



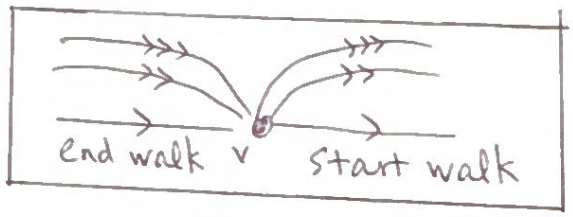
The remaining edges are not accessible via start/end vertices, but are accessible between "Day 1" and "Day 2" of our walk, so we fit these edges into our walk between those days using decimals between 1 and 2:



Def'n An Euler circuit is an Euler walk from v to v.

Thm A connected graph has an Euler circuit $\iff$ all its degrees are even.

Proof $\Rightarrow$ Same as before, but now v is both start & end vertex:



] passing thru uses even # of edges.

] uses 2 edges

So deg(v) is even as well.

Proof $\Leftarrow$ Do Hierholzer's algorithm, but pick any vertex to serve as both start & end vertex.

Why Euler walks useful? Efficient routes to plow snow, deliver mail, dispose garbage, check meters, patrol.